Horizons of stability

2008 J. Phys. A: Math. Theor. 41 244027

(http://iopscience.iop.org/1751-8121/41/24/244027)

J. Phys. A: Math. Theor. **41** (2008) 244027 (16pp) doi:10.1088/1751-8113/41/24/244027

Horizons of stability

Miloslav Znojil

Nuclear Physics Institute ASCR, 250 68 Řež, Czech Republic

E-mail: znojil@ujf.cas.cz

Received 8 October 2007, in final form 9 December 2007
Published 3 June 2008
Online at stacks.iop.org/JPhysA/41/244027

Abstract
Although the quantum bound-state energies may be generated by the so-called $\mathcal{PT}$-symmetric Hamiltonians $H = \mathcal{P}H^\dagger\mathcal{P} \neq H^\dagger$ where $\mathcal{P}$ is, typically, parity, the spectrum only remains real and observable (i.e., in the language of physics, the $\mathcal{PT}$-symmetry remains unbroken) inside a domain $\mathcal{D}$ of couplings. We show that the boundary $\partial\mathcal{D}$ (i.e., certain stability and observability horizon formed by Kato's exceptional points) remains algebraic (i.e., we determine it by closed formulae) for a certain toy-model family of N-dimensional anharmonic-oscillator-related matrix Hamiltonians $H^{(N)}$ with $N = 2, 3, \ldots, 11$.

PACS numbers: 03.65.BZ, 03.65.Ge, 03.65.Fd

1. Introduction

According to the abstract principles of quantum mechanics, the observable quantities (say, the spectra of energies $E_0 < E_1 < \cdots$ of bound states) should be constructed as eigenvalues of a certain self-adjoint operator $H = H^\dagger$ acting in some physical Hilbert space of states $\mathcal{H}$. Fortunately, the full and impressive generality of this formulation of the theory is rarely needed in its concrete applications. Most often, the Hilbert space is being chosen in its most common representation $\mathbb{L}_2(\mathbb{R})$ with elements $|\psi\rangle$ representing the square-integrable complex functions of a single variable x interpreted as a coordinate of a (quasi)particle.

The most common version of the Hamiltonian H composed of its kinetic and potential-energy parts leads to the constructions of the energies via a suitable phenomenological potential $V(x)$ entering the ordinary differential Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x). \tag{1}$$

In 1998, Bender and Boettcher [1] demonstrated that the spectra $\{E_n\}$ can remain real, amazingly, even if the potentials themselves become complex. This attracted the attention of physicists of different professional orientations ranging from supersymmetry [2] and field theory [3] to cosmology [4] and even magnetohydrodynamics [5].

Although Bender's and Boettcher's observation has been supported by many subsequent studies and concrete examples [6–8] it still looked like a paradox as it immediately implied that $H \neq H^\dagger$ in $\mathbb{L}_2(\mathbb{R})$. Fortunately, the resolution of the paradox proved rather easy [9]. It was sufficient to imagine that the Hamiltonian H can remain tractable as self-adjoint in *another* Hilbert space of states $\mathcal{H} \neq \mathbb{L}_2(\mathbb{R})$. A closely related *historical* paradox is that the usefulness of transfer of $H \neq H^\dagger$ to another Hilbert space $\mathcal{H}$ (where it becomes self-adjoint) has already been known and even applied successfully in nuclear physics in 1992 [10]. Still, it took time to clarify this parallelism [11] (cf also the comprehensive review paper [12] for further details).

On the level of the elementary ordinary differential example (1), one of the most important formal observations has been made in [6]. Its authors Dorey, Dunning and Tateo noted and described a *deep nontriviality* of the problem of the reality of the spectrum $\{E_n\}$. After they added some other free parameters (the physical meaning of which is not too relevant for our forthcoming argumentation) and after they replaced Bender's and Boettcher's $H^{(BB)} = p^2 + x^2(\mathrm{i}x)^\delta$ with $\delta \geqslant 0$ by a two-parametric family of their generalized Hamiltonians $H^{(\mathrm{DDT})}(\alpha, \ell)$, they revealed that the reality of the spectrum $\{E_n(\alpha, \ell)\}$ only takes place *inside a certain domain* of parameters (let us denote it as $\mathcal{D}(\alpha, \ell)$) with a highly nontrivial, spiked shape of its boundary $\partial\mathcal{D}(\alpha, \ell)$.

We found the latter observation extremely challenging, important, interesting and inspiring. The very existence of a finite boundary of the domain $\mathcal{D}(\alpha, \ell) \neq \mathbb{R}^2$ represents one of the main differences of the model from the differential Schrödinger equation (1) which are self-adjoint in the standard Hilbert space $\mathbb{L}_2(\mathbb{R})$ and which require that the potential $V(x)$ remains real. In the new framework, the change of space $\mathbb{L}_2(\mathbb{R}) \to \mathcal{H}$ implies that only the new potentials $V(x)$ *with parameters inside* $\mathcal{D}(\alpha, \ell)$ may consistently be chosen as complex [12].

The main motivation of the detailed study of the domains exemplified by $\mathcal{D}(\alpha, \ell)$ lies in their obvious practical relevance. Virtually no problems emerge in the current textbook scenario where the explicit specification of the physical domain of parameters $\mathcal{D}$ is usually trivial. The standard choice of the space $\mathbb{L}_2(\mathbb{R})$ and of a manifestly self-adjoint differential Hamiltonian $H = H^\dagger$ (depending on some J real couplings or other free parameters) usually enables us to work with the elementary, unrestricted domain $\mathcal{D} \equiv \mathbb{R}^J$ of these parameters. In contrast, the necessary determination of $\partial\mathcal{D}$ (which plays the role of a certain horizon of the observability and of the stability of the quantum system) proved fairly difficult even in the comparatively elementary example $H^{(\mathrm{DDT})}(\alpha, \ell) \neq [H^{(\mathrm{DDT})}(\alpha, \ell)]^\dagger$ as studied in [6].

In the widespread terminology coined in [1, 9] and employed also in the review [12], the set $\mathcal{D}(\alpha, \ell)$ of parameters could be called the domain of the so-called unbroken $\mathcal{PT}$-symmetry of the system in question. The scope of this specification is adapted to the models (1)—usually, one stays inside $\mathbb{L}_2(\mathbb{R})$ and specifies $\mathcal{P}$ as the parity reversal and $\mathcal{T}$ as the time reversal. Then one requires the $\mathcal{PT}$-symmetry of the Hamiltonians H which means that $\mathcal{PT}H = H\mathcal{PT}$ [1, 13]. The choice of the parameters inside $\mathcal{D}$ (i.e., the physical requirement of the measurability and reality of the spectrum) is then, finally, rephrased as the so-called $\mathcal{PT}$-symmetry of the wavefunctions (the review [12] can and should be consulted for all these details).

The nontriviality and the non-smooth, spiked shape of the curve $\partial\mathcal{D}(\alpha, \ell)$ as found in [6] opens the question of a generic characterization of the geometry of $\partial\mathcal{D}$ in the less specific setting. In what follows, we intend to review, extend and *complete* our results in this direction as published in the papers [14–21]. In these papers, we tried to bridge the apparent gap between the methods aimed at differential Schrödinger operators and at their alternative matrix representations with $H \neq H^\dagger$. A supplementary though still sufficiently appealing physical background of our series of studies has been found in the anharmonic-oscillator problem with the specific and popular $H = H^{(\mathrm{AHO})} \neq H^\dagger$ where $V(x) = V^{(\mathrm{AHO})}(x) = x^2 + \mathrm{i}gx^3$ or where $V(x)$ has been generalized to an arbitrary real polynomial in the purely imaginary variable ix.

The results of [14–21] (with particular attention paid to the constructions of the domains $\mathcal{D}$) will be reviewed here in sections 2 and 3. Their extension and completion will be described in sections 4 and 5. Section 6 is a summary.

2. Matrix models with small dimensions

In the review [10] written in the context of the so-called interacting boson models in nuclear physics, Scholtz *et al* emphasized that once a *non-unitary* Dyson's fermion-to-boson mapping is used, the calculation of the energies *gets simplified* while the price to be paid still proves reasonable for matrices. Indeed, in order to restore the Hermiticity of the Hamiltonians, just the *finite-dimensional* physical Hilbert space $\mathcal{H}$ had to be reconstructed *ad hoc*. This observation of merits of working with finite-dimensional matrices also formed a key encouragement of our forthcoming considerations.

2.1. Variational inspiration: truncated Hamiltonians

By assumption, our starting point given by Schrödinger equation (1) with $H = H^{(\mathrm{AHO})}$ remains simple and tractable in the usual Hilbert space $\mathbb{L}_2(\mathbb{R})$. At the same time, the necessary transition to the physical Hilbert space $\mathcal{H}$ (where our complex anharmonic-oscillator $H^{(\mathrm{AHO})}$ becomes self-adjoint) may remain complicated (interested readers can find all details in the literature cited in [12]). In such a situation, the use of the basis (*plus its subsequent variational truncation*) has been recommended in all the series of our studies [14–21]. In this way we were able to reduce the differential-operator Hamiltonians $H^{(\mathrm{AHO})}$ in equation (1) to the sequence of their partitioned matrix approximate forms

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \equiv \left(\begin{array}{ccc|ccc} H_{1,1} & \dots & H_{1,K} & H_{1,K+1} & \dots & H_{1,K+K'} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ H_{K,1} & \dots & H_{K,K} & H_{K,K+1} & \dots & H_{K,K+K'} \\ \hline H_{K+1,1} & \dots & H_{K+1,K} & H_{K+1,K+1} & \dots & H_{K+1,K+K'} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ H_{K+K',1} & \dots & H_{K+K',K} & H_{K+K',K+1} & \dots & H_{K+K',K+K'} \end{array}\right).$$

The truncated basis $\{|n, \pm\rangle\}$ has been numbered by the excitation quantum numbers $n = 0, 1, \dots$ and it has been partitioned with respect to the parity $\pm$. For our special anharmonic cubic $H^{(\mathrm{AHO})}$ this implied that A and D were real and diagonal, with $A_{k,k} = 4k+1$ and $D_{k,k} = 4k+3$ in suitable units. In parallel, all the necessary matrix elements of the purely imaginary matrices $B = \mathrm{i}B' = C^t = \mathrm{i}(C')^t$ (where ^{t} means transposition) have to be evaluated numerically. This is partially simplified by the observation that we may restrict our attention to the purely real matrices since

$$\left(\begin{array}{c|c} A & B \\ C & D \end{array}\right) \left(\begin{array}{c} u \\ \hline v \end{array}\right) \equiv \left(\begin{array}{c|c} A & B' \\ \hline -C' & D \end{array}\right) \left(\begin{array}{c} u \\ \hline \mathrm{i}v \end{array}\right).$$

Unfortunately, our attempts to construct the domains $\mathcal{D}$ in closed form were only successful at $N = K + K' = 2$ [14], at $N = K + K' = 3$ [16] and, with certain surmountable difficulties, at $N = K + K' = 4$ [17].

2.2. Tridiagonal chain models in the strong-coupling regime

The latter results helped us to imagine that the variations of the vast majority of the (real) matrix elements of $B' = (C')^t$ did not lead to any really significant changes in the overall structure of the respective boundaries $\partial\mathcal{D}^{(N)}$. For this reason we further re-arranged the basis and reduced the variability of the submatrices B' and C' by setting many of their matrix elements equal to zero. As a net result of all these more or less natural simplifications we arrived, in [18], at another family of the matrix toy models exhibiting the general tridiagonal structure

$$H^{(N)} = \begin{bmatrix} -(N-1) & g_1 & & & & \\ -g_1 & -(N-3) & g_2 & & & \\ & -g_2 & \ddots & \ddots & & \\ & & \ddots & N-5 & g_2 & \\ & & & -g_2 & N-3 & g_1 \\ & & & & -g_1 & N-1 \end{bmatrix}. \tag{2}$$

At the dimensions $N = 2J$ or $N = 2J + 1$ these models depend just on a J-plet of real couplings $g_1, g_2, \ldots, g_J$. One of the most important formal merits of these models is that at any dimension N, all the domain $\mathcal{D}^{(N)}$ lies inside a finite hypercube [18]. The boundary $\partial\mathcal{D}^{(N)}$ itself can be characterized by its strong-coupling maxima which were obtained in the following closed form:

$$g_n^{(\max)} = \pm(N-n)n, \qquad n = 1, 2, \ldots, J. \tag{3}$$

Although the strong-coupling result (3) looks easy, its derivation required extensive computer-assisted symbolic manipulations. Via a nontrivial extrapolation guesswork we revealed that geometrically, the horizons $\partial\mathcal{D}^{(N)}$ are (hyper)surfaces with protruded spikes called extreme exceptional points, EEPs. This intuitive picture has been complemented by the more quantitative descriptions of $\partial\mathcal{D}^{(N)}$ in [19, 20]. It was based on the strong-coupling perturbation ansatz using an auxiliary, formally redundant small parameter t,

$$g_n = g_n^{(\max)}\sqrt{(1-\gamma_n(t))}, \qquad \gamma_n(t) = t + t^2 + \cdots + t^{J-1} + G_n t^J. \tag{4}$$

This ansatz extrapolates the rigorous $J \leqslant 2$ fine-tuning rules as derived in [14, 18] to all J.

2.3. Secular equations

Once we choose $N = 2J$ or $N = 2J + 1$, abbreviate $E^2 = s$ and, at all the odd dimensions $N = 2J + 1$, ignore the persistent energy level $E_J^{(2J+1)} = 0$, we find that all the secular equations $\det(H^{(N)} - E) = 0$ have the same polynomial form,

$$s^J - \binom{J}{1} s^{J-1}P + \binom{J}{2} s^{J-2}Q - \binom{J}{3} s^{J-3}R + \cdots = 0. \tag{5}$$

At all J and N, the coefficients $P, Q, R, \ldots$ should be understood as real polynomial functions of the J-plets of squares g_k^2 of our real matrix elements. Once all the energies are assumed real (i.e., equivalently, once all the roots s_k of equation (5) happen to be non-negative), we immediately deduce the following relations tractable as necessary conditions imposed upon our coefficients in (5),

$$\begin{aligned}
&\binom{J}{1} \cdot P = s_1 + s_2 + \cdots + s_J \geqslant 0,\\
&\binom{J}{2} \cdot Q = s_1 s_2 + s_1 s_3 + \cdots + s_1 s_J + s_2 s_3 + s_2 s_4 + \cdots + s_{J-1} s_J \geqslant 0,\\
&\binom{J}{3} \cdot R = s_1 s_2 s_3 + s_1 s_2 s_4 + \cdots + s_{J-2} s_{J-1} s_J \geqslant 0,\\
&\ldots .
\end{aligned} \tag{6}$$

In the opposite direction, the set of the necessary inequalities $P \geqslant 0$, $Q \geqslant 0, \ldots$ is incomplete as it does not provide the desirable sufficient condition of observability. It admits complex roots s in general (take a sample secular polynomial $(s^2 + 1)(s - 2)$ for illustration). This shows that our problem of the determination of the physical domains $\mathcal{D}^{(N)}$ of couplings is mathematically nontrivial even at the smallest dimensions N and J.

For a given prototype Hamiltonian $H^{(N)}$ and under the constraints (6), the determination of the domain $\mathcal{D}^{(N)} = \mathcal{D}(H^{(N)})$ is *equivalent* to the guarantee of the non-negativity of all the J roots s_k of equation (5). Keeping this idea in mind, the explicit forms of the corresponding sufficient conditions are to be given here for the first ten smallest matrix dimensions $N = 2, 3, \ldots, 11$.

3. The domains $\mathcal{D}^{(2J)}$ and $\mathcal{D}^{(2J+1)}$: a brief review of the known results

3.1. Methodical inspiration: the non-negativity of the root of equation (5) at $J = 1$

The first nontrivial illustration of the current *Hermitian* Schrödinger's bound-state problem is provided by the two-by-two real-matrix model

$$H|\psi\rangle = E|\psi\rangle, \qquad H = H(a, b, d) = \begin{pmatrix} a & b \\ b & d \end{pmatrix} = H^\dagger(a, b, d).$$

Its three-parametric spectrum is *always* real and, therefore, observable,

$$E = E_\pm(a, b, d) = \tfrac{1}{2}\big[a + d \pm \sqrt{(a - d)^2 + 4b^2}\big].$$

For $H = H(a, b, d) = H^\dagger$ the three-dimensional physical domain $\mathcal{D}(a, b, d)$ of parameters giving real spectra coincides with *all* $\mathbb{R}^3$.

The parallel $\mathcal{PT}$-symmetric two-by-two example is very similar,

$$H = H'(a, b, d) = \begin{pmatrix} a & b \\ -b & d \end{pmatrix}, \qquad E = E'_\pm(a, b, d) = \frac{1}{2}\big[a + d \pm \sqrt{(a - d)^2 - 4b^2}\big].$$

For each individual choice of the parameters a, b and d, the reality of the spectrum $E'_\pm(a, b, d)$ of the primed Hamiltonian $H'(a, b, d)$ is fragile and it must be guaranteed and proved at a given triplet of parameters. The reality and stability of the primed system can only be achieved inside a *perceivably smaller* domain $\mathcal{D}'(a, b, d)$ with the easily specified EP horizon,

$$\partial\mathcal{D}'(a, b, d) = \{(a, b, d) \in \mathbb{R}^3 \mid (a - d)^2 = 4b^2\}.$$

The interior of the non-compact manifold $\mathcal{D}'(a, b, d)$ is specified by the single elementary constraint $b \in (-|a - d|, |a - d|)$. This may be interpreted as a fact that the variability of the parameters a and d is entirely redundant for *qualitative* considerations. It makes sense to get rid of them by the multiplicative re-scaling of all the parameters and by the subsequent shift of the energy scale leading to the generic choice of $a = -1$ and $d = 1$.

In the context of section 2.3, the latter reduction leads us to the linear version $s - P = 0$ of secular equation (5) at $J = 1$ which has the single root $s_0 = P$. The non-negativity of this root is equivalent to the non-negativity of the coefficient P. In terms of the single coupling $g_1 = a$ available at $J = 1$, the necessary and sufficient criteria of the observability of $H^{(2)} = H^{(2)}(a)$ or $H^{(3)} = H^{(3)}(a)$ read $P^{(2)}(a) = 1 - a^2 \geqslant 0$ and $P^{(3)}(a) = 4 - 2a^2 \geqslant 0$, respectively. In a way transferable to any dimension, the explicit definitions $\mathcal{D}^{(2)}(a) = (-1, 1)$ and $\mathcal{D}^{(3)}(a) = (-\sqrt{2}, \sqrt{2})$ may be re-read as definitions of the corresponding EP horizons $\partial\mathcal{D}^{(2)}(a) = \{-1, 1\}$ and $\partial\mathcal{D}^{(3)}(a) = \{-\sqrt{2}, \sqrt{2}\}$.

In the language of phenomenology, one notes an important complementarity between the parameter-dependence of the toy spectra $E_\pm(a, b, d)$ and $E'_\pm(a, b, d)$ at $N = 2$. In the former example, *all* of the energies $E_\pm(a, b, d)$ remain safely real. The second, primed model is less easy to deal with. There exists the whole set of the eligible two-by-two metric operators $\Theta = \Theta^\dagger > 0$ which define the inner product in the corresponding two-dimensional toy Hilbert space $\mathcal{H}'$ (cf [15]). Thus, the operator $H'(a, b, d)$ represents an observable, in $\mathcal{H}'$, in spite of its manifest non-Hermiticity in the auxiliary two-dimensional Hilbert space $\mathcal{H}$ (where the metric is the Dirac's simplest identity operator). With parameters inside $\mathcal{D}'(a, b, d)$ the primed model remains safely compatible with all the postulates of quantum mechanics, therefore.

3.2. The non-negativity of all the roots of equation (5) at $J = 2$

At $J = 2$ the quadratic version $s^2 - 2Ps + Q = 0$ of secular equation (5) has two roots $s_\pm = P \pm \sqrt{P^2 - Q}$. These two roots remain real if and only if $B \equiv P^2 - Q \geqslant 0$. In the subdomain of parameters where $B \geqslant 0$ they remain both non-negative if and only if $P \geqslant 0$ and $Q \geqslant 0$. We can summarize that the required sufficient criterion reads

$$P \geqslant 0, \qquad P^2 \geqslant Q \geqslant 0. \tag{7}$$

In an alternative approach, *without* an explicit reference to the available formula for $s_\pm$, let us contemplate the parabolic curve $y(s) = s^2 - 2Ps$ which remains safely positive, in the light of our assumption (6), at all the negative $s < 0$. This curve can only intersect the horizontal line $z(s) = -Q$ at some non-negative points $s \geqslant 0$.

The proof of non-negativity of all the roots of our secular equation degenerates to the proof that there exist two real points of intersection of the $J = 2$ parabola $y(s)$ with the horizontal line $z(s)$ (which lies below zero) at some $s \geqslant 0$. Towards this end we consider the minimum of the curve $y(s)$ which lies at the point s_0 such that $y'(s_0) = 0$, i.e., at $s_0 = P$. This minimum must lie *below* (or, at worst, at) the horizontal line of $z(s) = -Q \leqslant 0$. But the minimum value of $y(s_0)$ is known, $y(P) = -P^2$. Thus, the condition of intersection $y(s_0) \leqslant z(s_0)$ gives the formula $P^2 \geqslant Q$.

It is amusing to note that once equation (6) holds, the inequality $P^2 - Q \geqslant 0$ is equivalent to the reality of the roots simply because $P^2 - Q \equiv (s_1 - s_2)^2/4$. Even for some other two-parametric matrices, precisely this type of requirement is responsible for an important part of the EP boundary $\partial\mathcal{D}$ (cf [16, 17] for details).

4. The domains $\mathcal{D}^{(2J)}$ and $\mathcal{D}^{(2J+1)}$: new results

The determination of the physical horizons $\partial\mathcal{D}^{(N)}$ of our models $H^{(N)}$ becomes a more or less purely numerical task at the very large matrix dimensions N [20]. In an opposite extreme, as we already noted, the non-numerical exceptions have been found at $N = 2$ [14], at $N = 3$ [16] and at the next two dimensions $N = 4$ and $N = 5$ [17, 18]. Now we intend to complement these observations by showing that the closed-form constructions of the prototype horizons $\partial\mathcal{D}^{(N)}$ remain feasible up to the dimension as high as $N = 11$.

4.1. The non-negativity of all the roots of equation (5) at $J = 3$

Neither at $N = 6$ nor at $N = 7$ the sufficient condition of non-negativity of all the energy roots s is provided by the three necessary rules $P \geqslant 0$, $Q \geqslant 0$ and $R \geqslant 0$ of equation (6). Let us return, therefore, to the second method used in paragraph 3.2 and derive another inequality needed as a guarantee of the reality of the energies. In the first step one notes that all the three components of the polynomial

$$y(s) = s^3 - 3Ps^2 + 3Qs = R, \qquad J = 3$$

remain safely non-positive at $s < 0$. Whenever the roots are guaranteed real, their non-negativity $s_n \geqslant 0$ with $n = 1, 2, 3$ is already a consequence of the three constraints (6). A guarantee of their reality is less trivial but it still can be deduced from the shape of the function $y(s)$ on the half-axis $s \geqslant 0$, i.e., from the existence and properties of a real maximum of $y(s)$ (at $s = s_-$) and of its subsequent minimum (at $s = s_+$). At both these points the derivative $y'(s) = 3s^2 - 6Ps + 3Q$ vanishes so that both the roots $s_\pm = P \pm \sqrt{P^2 - Q}$ of $y'(s)$ must be real and non-negative. This condition is always satisfied for the *real* roots s_k of $y(s)$ since

$$B = P^2 - Q \equiv \tfrac{1}{54}[(s_1 + s_2 - 2s_3)^2 + (s_2 + s_3 - 2s_1)^2 + (s_3 + s_1 - 2s_2)^2] \geqslant 0.$$

In the next step, the sufficient condition of the reality of the roots s_k will be understood as equivalent to the doublet of the inequalities $y(s_-) \geqslant R$ and $y(s_+) \leqslant R$. Here we may insert $s_\pm^2 = 2Ps_\pm - Q$ and get the two inequalities which are more explicit,

$$2(P^2 - Q)s_- \leqslant PQ - R \leqslant 2(P^2 - Q)s_+. \tag{8}$$

They restrict the range of a new symmetric function of the roots,

$$PQ - R \equiv \tfrac{1}{9}[s_1s_2(s_1 + s_2 - 2s_3) + s_2s_3(s_2 + s_3 - 2s_1) + s_3s_1(s_3 + s_1 - 2s_2)].$$

After another insertion of the known $s_\pm$ we arrive at a particularly compact formula

$$2(P^2 - Q)^{3/2} \geqslant R - 3PQ + 2P^3 \geqslant -2(P^2 - Q)^{3/2}$$

or, equivalently,

$$4(P^2 - Q)^3 \geqslant (R - 3PQ + 2P^3)^2.$$

Due to the numerous cancellations the latter relation further degenerates to the most compact missing necessary condition

$$3P^2Q^2 + 6RPQ \geqslant 4Q^3 + R^2 + 4RP^3. \tag{9}$$

Our task is completed. In combination with equations (6), equation (9) plays the role of the guarantee of the reality of the energy spectrum.

4.2. The non-negativity of all the roots of equation (5) at $J = 4$

In a search for the non-negative roots of the quartic secular equation

$$\det(H^{(8,9)} - EI) = x^4 - 4Px^3 + 6Qx^2 - 4Rx + S \equiv y(x) + S = 0 \tag{10}$$

we note that all the four N-dependent coefficients P, Q, R and S evaluate as polynomials in the squares of the four coupling parameters g_k, $k = 1, 2, 3, 4$. Once all these four expressions are kept non-negative, the curves $y(x)$ and $z(x) = -S$ do not intersect at $x < 0$. At $x \geqslant 0$ they do intersect four times at $x \geqslant 0$ (as required), provided only that the three extremes of $y(x)$

can be found at the three non-negative real roots $x_{1,2,3}$ of the extreme-determining equation

$$y'(x_{1,2,3}) = 4\big(x_{1,2,3}^3 - 3Px_{1,2,3}^2 + 3Qx_{1,2,3} - R\big) = 0. \tag{11}$$

In an ordering $0 \leqslant x_1 \leqslant x_2 \leqslant x_3$ of these roots we arrive at the three sufficient conditions

$$y(x_1) \leqslant -S, \qquad y(x_2) \geqslant -S, \qquad y(x_3) \leqslant -S \tag{12}$$

guaranteeing that the parameters lie inside $\mathcal{D}^{(8)}$ or $\mathcal{D}^{(9)}$.

All the three quantities x_k satisfy the cubic equation $y'(x) = 0$ so that its premultiplication by x enables us to single out the fourth powers of the roots,

$$x_{1,2,3}^4 = 3Px_{1,2,3}^3 - 3Qx_{1,2,3}^2 + Rx_{1,2,3}.$$

In the other words, we can eliminate all the fourth powers of these roots from $y(x_{1,2,3})$. This simplification reduces all the three items in equation (12) to the three polynomial inequalities of the third degree,

$$-Px_{1,3}^3 + 3Qx_{1,3}^2 - 3Rx_{1,3} + S \leqslant 0,$$
$$-Px_2^3 + 3Qx_2^2 - 3Rx_2 + S \geqslant 0.$$

Repeating the same elimination of the maximal powers once more, we may insert $x_{1,2,3}^3 = 3Px_{1,2,3}^2 - 3Qx_{1,2,3} + R$ and arrive at another equivalent triplet of inequalities

$$-Bx_1^2 + 2B^{3/2}Cx_1 \leqslant B^2D, \tag{13}$$
$$-Bx_2^2 + 2B^{3/2}Cx_2 \geqslant B^2D, \tag{14}$$
$$-Bx_3^2 + 2B^{3/2}Cx_3 \leqslant B^2D. \tag{15}$$

The old abbreviations $B = P^2 - Q$ and $2B^{3/2}C = PQ - R$ plus a new one, $3B^2D = PR - S$ enable us to define $Y_{1,2,3} := x_{1,2,3,}/\sqrt{B}$. This yields our final triplet of the very transparent quadratic-equation conditions

$$Y_1^2 - 2CY_1 + D \geqslant 0, \tag{16}$$
$$Y_2^2 - 2CY_2 + D \leqslant 0, \tag{17}$$
$$Y_3^2 - 2CY_3 + D \geqslant 0. \tag{18}$$

The auxiliary roots $Y_\pm = C \pm \sqrt{C^2 - D}$ must be real and non-negative so that we must guarantee that $D \geqslant 0$ and $C^2 \geqslant D$. The conclusion is that equations (16)–(18) degenerate to the four elementary requirements

$$Y_1 \leqslant Y_- \leqslant Y_2 \leqslant Y_+ \leqslant Y_3. \tag{19}$$

Together with the inequalities $B \geqslant 0$, $Q \geqslant 0$ and $-1 \leqslant C - \sqrt{1 + Q/B} \leqslant 1$ they form the final and complete algebraic definition of the domains $\mathcal{D}^{(8)}$ and $\mathcal{D}^{(9)}$.

We can summarize that at $J = 4$, the feasibility of the non-numerical construction of the domains $\mathcal{D}^{(8)}$ and $\mathcal{D}^{(9)}$ (determined by equations (6) and (19)) is based on the non-numerical solvability of the third-order polynomial equation (11).

4.3. The non-negativity of all the roots of equation (5) at $J = 5$

Let us finally proceed to $H^{(N)}$ with $N = 10$ and/or $N = 11$ which leads to the secular equations of the fifth degree,

$$x^5 - 5Px^4 + 10Qx^3 - 10Rx^2 + 5Sx - T \equiv y(x) - T = 0. \tag{20}$$

From our present point of view the problem of the construction of the respective horizons $\partial\mathcal{D}^{(N)}$ remains solvable exactly since the derivative $y'(x)$ is still a polynomial of the mere fourth degree,

$$\tfrac{1}{5}y'(x) = x^4 - 4Px^3 + 6Qx^2 - 4Rx + S. \tag{21}$$

The exact, real and non-negative values $x_1 \leqslant x_2 \leqslant x_3 \leqslant x_4$ of the four roots of $y'(x)$ (which determine the extremes of the function $y(x)$) may still be considered available in the closed form.

In a way which parallels our preceding considerations we may assume that the five N-dependent non-negative coefficients $P \geqslant 0$, $Q \geqslant 0$, $R \geqslant 0$, $S \geqslant 0$ and $T \geqslant 0$ obey also all the additional inequalities derived in the preceding sections. We may then treat our secular problem (20) as a search for the graphical intersections between the (non-negative) constant curve $z(x) = T$ and the graph of the polynomial $y(x)$ of the fifth degree (which can only be non-negative at $x \geqslant 0$).

Inside the domain $\mathcal{D}^{(N)}$, the quintuplet of the (unknown but real and non-negative) physical energy roots x_a, x_b, x_c, x_d and x_e has to obey the obvious intertwining rule

$$0 \leqslant x_a \leqslant x_1 \leqslant x_b \leqslant x_2 \leqslant x_c \leqslant x_3 \leqslant x_d \leqslant x_4 \leqslant x_e.$$

The way towards the sufficient condition of the existence of the real energy spectrum remains the same as above, requiring

$$y(x_1) \geqslant T, \qquad y(x_2) \leqslant T, \qquad y(x_3) \geqslant T, \qquad y(x_4) \leqslant T. \tag{22}$$

The lowering of the degree should reduce equation (22) to the quadruplet

$$w(Y_1) \leqslant 0, \qquad w(Y_2) \geqslant 0, \qquad w(Y_3) \leqslant 0, \qquad w(Y_4) \geqslant 0, \tag{23}$$

where the re-scaling $x_{1,2,3,4} = Y_{1,2,3,4}\sqrt{B}$ applies to the arguments of the brand new auxiliary polynomial function of the third degree in Y,

$$w(Y) = Y^3 - 3CY^2 + 3DY - G.$$

Besides the same abbreviations as above, we introduced here a new one, for $PS - T \equiv 4B^{5/2}G$. The new and specific problem now arises in connection with the necessity of finding the three auxiliary and, of course, real and non-negative roots of the cubic polynomial $w(Y)$. Once we mark them, in the ascending order, by the Greek-alphabet subscripts, we should either postulate our (in principle, explicit) knowledge of their real and non-negative values $Y_\alpha \leqslant Y_\beta \leqslant Y_\gamma$ or, in another perspective, we may recollect simply the above-derived conditions which restrict the range of the three coefficients C, D and G in the cubic polynomial $w(Y)$.

We may immediately conclude that the last feasible specification of the domains $\mathcal{D}^{(10)}$ and $\mathcal{D}^{(11)}$ will be given by the following set of the inequalities:

$$Y_1 \leqslant Y_\alpha \leqslant Y_2 \leqslant Y_\beta \leqslant Y_3 \leqslant Y_\gamma \leqslant Y_4. \tag{24}$$

This is the desired set of the missing algebraic formulae which complete the sufficient condition of the reality of the spectra. We can emphasize that at $J = 5$, the feasibility of the present non-numerical constructions of the most complicated five-dimensional though still algebraic domains $\mathcal{D}^{(N)}$ is related again to the most complicated though still non-numerical solvability of the extreme-determining fourth-order polynomial equation (21).

The series of the solvable models is, obviously, exhausted. Any attempted extension of the recipe beyond $N = 11$ would suffer from the necessity of using mere numerical auxiliary functions of couplings g_k representing the roots of the extreme-determining higher-order polynomials.

5. The wedges of the hypersurfaces $\partial\mathcal{D}^{(N)}$

The existence of the algebraic formulae which determine all the boundaries $\partial\mathcal{D}^{(N)}$ up to $N = 11$ opens a way towards a verification of the strong-coupling perturbation results of [18] and [19]. Hopefully, some other similar qualitative or geometric features of the observability horizons $\partial\mathcal{D}$ assigned to a given $\mathcal{PT}$-symmetric Hamiltonian H will be also detected or discovered via the reduction of the model to a series of its N by N approximations of the prototype form $H^{(N)}$.

In an alternative setting, we can fix the dimension N and search for the generic features represented by the given model $H^{(N)}$. Moving beyond the perturbative framework, for example, our ansatz (4) could be then re-interpreted as a *precise* change of the variables in the space of our free dynamical parameters. The redundant measure of smallness t may be now fixed arbitrarily. Instead of the couplings g_k one can also decide to work with the free parameters γ_k or even with G_ks. For illustration one may recollect the two-by-two Hamiltonian

$$H^{(2)} = \begin{pmatrix} -1 & \sqrt{1-\alpha} \\ -\sqrt{1-\alpha} & 1 \end{pmatrix}, \qquad \alpha \in (0, 1) \tag{25}$$

with the two-point spectrum $E_{\pm}^{(2)} = \pm\sqrt{\alpha}$. The variability of the new parameter coincides with $\mathcal{D}^{(2)}(\alpha) \equiv (0, 1)$ since there are no additional constraints.

At the higher dimensions N, our horizons $\partial\mathcal{D}^{(N)}$ may be shown to exhibit a generic hedge-hog-like shape as well as certain reflection symmetries. They allow us to restrict our attention to the subdomains of $\mathcal{D}^{(N)}$ with the positive g_k s, i.e., with the real quantities $\gamma_J^{(N)} = \alpha, \gamma_{J-1}^{(N)} = \beta, \ldots$ which should all remain non-negative and smaller than one, $\gamma_k^{(N)} \in (0, 1)$.

5.1. *New forms of approximations*

Let us now return to the description of the structure of the boundaries of the domains $\mathcal{D}^{(N)}$ at $N = 6$ and $N = 7$ by means of our key inequality (8). In this form of the rigorous guarantee of the reality of the energies at $J = 3$ we may set $P^2 = P^2(B, Q) = B + Q$, postulate $B \geqslant 0$, $Q \geqslant 0$ and insert

$$s_{\pm} = \sqrt{B+Q} \pm \sqrt{B} \geqslant 0$$

in equation (8). With an abbreviation $C := (PQ - R)/(2B^{3/2}) \geqslant 0$ this gives the pair of inequalities $s_- \leqslant C\sqrt{B} \leqslant s_+$ or, equivalently,

$$\sqrt{1+q} - 1 \leqslant C \leqslant \sqrt{1+q} + 1, \qquad q = \frac{Q}{B} \in (0, \infty). \tag{26}$$

Once we note that $PQ \equiv Q\sqrt{B+Q} = qB^{3/2}\sqrt{1+q}$ we may return from the auxiliary C to the original R and rewrite equation (26) as our final, perceivably simplified one-parametric constraint imposed upon the allowed range of the variability of the value of the polynomial R,

$$1 + \left(\frac{q}{2} - 1\right)\sqrt{1+q} \geqslant \frac{R}{2B^{3/2}} \geqslant \begin{cases} 0, & q \leqslant 3, \\ \left(\frac{q}{2} - 1\right)\sqrt{1+q} - 1, & q > 3. \end{cases} \tag{27}$$

In such a reparametrization of the physical domains $\mathcal{D}^{(6,7)}$ we use $B \geqslant 0$ and $Q \geqslant 0$ and employ our final form of the two-sided inequality (27) as a definition of the allowed range of the remaining quantity R. The latter inequality if particularly strong at the smallest ratios $q = Q/B$. As long as

$$1 + \left(\frac{q}{2} - 1\right)\sqrt{1+q} = \frac{3}{8}q^2 - \frac{1}{8}q^3 + \frac{9}{128}q^4 - \frac{3}{64}q^5 + \frac{35}{1024}q^6 + \mathcal{O}(q^7),$$

we see that the smallness of q implies the *second-order* smallness of R. This means that in the regime where $Q \ll B$, the three-dimensional physical domains $\mathcal{D}^{(6,7)}$ are very narrow in their third dimension represented by R.

5.2. *The pairwise confluences of the levels at* $J = 3$

For the sake of definiteness let us choose just $N = 6$ and abbreviate

$$g_1 = c \leqslant \sqrt{5}, \qquad g_2 = b \leqslant 2\sqrt{2}, \qquad g_3 = a \leqslant 3$$

in the six-by-six version of matrix (2). It is easy to deduce that the domain $\mathcal{D}^{(6)}$ is circumscribed by the ellipsoidal surface given by the equation

$$P = -(a^2 + 2b^2 + 2c^2 - 35)/3 = 0.$$

The other two obvious constraints read

$$3Q = b^4 + 2c^2a^2 - 44b^2 + 28c^2 - 34a^2 + c^4 + 259 + 2b^2c^2 \geqslant 0$$

and

$$-R = a^2c^4 - 10b^2c^2 + 30c^2a^2 + 225a^2 - 30c^2 - c^4 - 25b^4 - 225 - 150b^2 \geqslant 0.$$

The last constraint needed to define $\mathcal{D}^{(6)}$ is then given by equation (8).

For illustrative purposes, the latter, purely algebraic description of the geometric shape of the boundary $\partial\mathcal{D}^{(6)}$ may be complemented by the explicit evaluation of all the six energy levels (say, $E_0 \leqslant E_1 \leqslant \cdots \leqslant E_5$). The easiest answer is obtained at the EEP singular couplings where the complete confluence of all the energies takes place, $E_0 = E_1 = \cdots = E_5$.

A clear insight in the structure of the spectrum remains available in the small EEP vicinity as well. For example, the innermost pair of the energies E_2 and E_3 can coincide at $\partial\mathcal{D}^{(6)}$ (and, subsequently, complexify out of $\mathcal{D}^{(6)}$) while the remaining two doublets remain real, or vice versa. In the former scenario we encounter the pairwise coincidence of $E_2 = E_3 = 0$ at $\partial\mathcal{D}^{(6)}$. In the latter case we may abbreviate $E_0 = E_1 = -4z = -E_4 = -E_5$ and compute the unknown quantity $z \in (0, 1)$.

Alternatively, the two outermost levels (namely, $E_0 = -E_5 = -\sqrt{5y}$ where $y \in (0, 1)$) can stay real while the confluence only involves the two internal energy doublets at the shared values of $E_3 = E_4 = 2\sqrt{x} = -E_1 = -E_2$ where $x \in (0, 1)$. This is the most complicated example where one has the relation

$$(s - s_{\max})[s - s_{\min}]^2 = s^3 - (32x^2 + 25y^2)s^2 + (256x^4 + 800x^2y^2)s - 6400x^4y^2 = 0$$

which defines a sub-surface of $\partial\mathcal{D}^{(6)}$.

5.3. *The pairwise confluences of the exceptional points*

Let us now return to the $J = 3$ option where the surface of the three-dimensional domain $\mathcal{D}^{(6)}$ can be visualized as composed, locally, of the two smooth sub-surfaces which intersect along a certain double exceptional point (DEP) curve. In terms of the single free parameter z, the

DEP secular equation degenerates to the formula $E^2(E+4z)^2(E-4z)^2 = 0$ obtainable from equation (5),

$$s[s-(4z)^2]^2 = s^3 - 2(4z)^2 s^2 + (4z)^4 s = 0. \tag{28}$$

It is fortunate that the necessary analysis can still be performed non-numerically since equation (28) is easy to compare with the true secular equation (5) with coefficients given in section 5.2. As long as the factorizable coefficient at s^0 must vanish, we get the first DEP constraint

$$[ac^2 + 15a + (15 + c^2 + 5b^2)][ac^2 + 15a - (15 + c^2 + 5b^2)] = 0$$

so that we may eliminate

$$a = \pm\frac{15 + c^2 + 5b^2}{c^2 + 15}.$$

In the quadrant of the $a-b-c$ space with the positive a, the plus sign must be chosen,

$$a = 1 + \frac{5b^2}{c^2 + 15}.$$

Thus, we have $3 \geqslant a \geqslant 1$ in the closed formula for

$$b^2 = \tfrac{1}{5}(c^2 + 15)(a-1) \tag{29}$$

or, alternatively, for

$$c^2 = \frac{5b^2}{a-1} - 15.$$

This result is to be complemented by the other two relations

$$3Q(c,b,a) = 32z^2, \qquad R(c,b,a) = 128z^4.$$

A straightforward elimination of z^2 gives the second DEP condition

$$-66a^2 - 36b^2 + 4c^2a^2 - 189 + 252c^2 - 4b^2a^2 - a^4 = 0$$

with the two compact roots

$$a_\pm^2 = 2c^2 - 33 - 2b^2 \pm 2\sqrt{c^4 + 30c^2 - 2b^2c^2 + 225 + 24b^2 + b^4}.$$

The acceptable one must be non-negative. For a_-^2 this would mean that $2c^2 \geqslant 33 + 2b^2$ while, at the same time, $63 + 12b^2 \geqslant 84c^2$. These two conditions are manifestly incompatible so that we must accept the upper-sign root a_+^2 which is automatically positive for all the large $2c^2 \geqslant 33 + 2b^2$. It also remains positive for all the smaller c^2 constrained by the requirement

$$84c^2 \geqslant 63 + 12b^2.$$

After the insertion of the definition (29) of b^2 we arrive at the final formula

$$84c^2 \geqslant 63 + \tfrac{12}{5}(a-1)(15 + c^2). \tag{30}$$

It defines a manifestly non-empty domain of parameters at which one encounters the pairwise confluences of Kato's exceptional points.

6. Summary and outlook

There exist two reasons why our knowledge of the physical domains $\mathcal{D}$ is relevant. Firstly, their boundaries $\partial\mathcal{D}$ are marking the breakdown of the reality and observability of the spectra $\{E_n\}$. Secondly, these boundaries also represent a regime where the matrices $H^{(N)}$ cease to be diagonalizable. Thus, it is the *simultaneous* degeneracy of the energies and of the wavefunctions which characterizes the physics near the boundary $\partial\mathcal{D}$ of the dynamical domain where the quantum system starts to become unstable.

6.1. The onset of instabilities along $\partial\mathcal{D}$

In Landau's textbook [22] on quantum mechanics the emergence of an instability of a quantum system is exemplified by a particle in a strongly singular attractive potential $V(\vec{x}) = G/|\vec{x}|^2$. At the critical value $G_{(\min)} = -1/4$ of its strength one encounters a horizon beyond which the particle starts falling on the centre. Vice versa, the system remains stable and physical on all the interval $\mathcal{D} = (-1/4, \infty)$ of couplings G. From the pragmatic point of view Landau's example is not too well selected since the falling particle should release, hypothetically, an infinite amount of the energy during its fall. A slightly better textbook example of the loss of the stability is provided, therefore, by Dirac's electron which moves in a superstrong Coulomb potential. In the language of physics, particle–antiparticle pairs are created in the system beyond a critical charge ($Z_{(\max)} = 137$ in suitable units [23]).

What is shared by the above two *Hermitian* sample Hamiltonians is that they are well defined in a certain domain $\mathcal{D}$ of parameters while they lose sense and applicability for parameter(s) beyond certain horizon. On a less intuitive level, similar situations have been studied by Kato [24]. He considered several finite-matrix toy Hamiltonians $H(\lambda)$. He paid attention to the *unphysical, complex* values of λ and deduced that the related (in general, complex) spectra $E_n(\lambda)$ change smoothly with the variation of the parameter λ unless one encounters certain critical, exceptional points $\lambda^{(\mathrm{EP})}$.

In the above context, certain carefully selected *non-Hermitian* examples seem to be able to offer *the best* illustrative examples in the physical stability analysis. In addition, our low-dimensional non-Hermitian tridiagonal matrices also path the way to a combination of mathematics and physics. In particular, in the context of pure mathematics, our present set of the solvable examples $H^{(N)}$ with $N \leqslant 11$ enabled us to *construct and study* the *real* version of Kato's exceptional points $\lambda^{(\mathrm{EP})} \in \partial\mathcal{D}^{(N)}$. In parallel, the abundance of the J free parameters in our models looks more suitable for phenomenological purposes. In particular, we believe that a systematic characterization of a collapse of a realistic physical system could make use of these specific models offering an important link to the possible future classification and, perhaps, typology of quantum catastrophes [20].

6.2. Degeneracies along the horizons $\partial\mathcal{D}$

The obvious theoretical appeal of the problem of stability may be perceived as one of the explanations of the recent growth of popularity of the $\mathcal{PT}$-symmetric and, more generally, η-pseudo-Hermitian Hamiltonians with promising relevance in quantum field theory [12, 25] and in quantum physics in general [11, 26]. Although the origin of the latter ideas can be traced back to the very early days of quantum theory [25], the feasibility of its separate implementations have long been treated as a mere mathematical and/or physical curiosity (cf, e.g., [13, 27] for illustration).

One of the serious technical shortcomings of the $\mathcal{PT}$-symmetric and other similar models is that their spectra are real (i.e., observable) in domains $\mathcal{D}$ with, sometimes, very complicated and strongly Hamiltonian-dependent shape of their EP boundaries $\partial\mathcal{D}$. For an uninterrupted development of their study it may prove very fortunate that an explicit analytic description of the horizons $\partial\mathcal{D}$ has been shown available here for all the matrix $\mathcal{PT}$-symmetric chain models $H^{(N)}$ with $N \leqslant 11$.

In this context, it is particularly important that several recent microwave measurements [28] confirmed the observability of the abstract Kato's exceptional points $\lambda^{(EP)}$ in practice. These experiments re-attracted attention to the theoretical analyses of the EP horizons, say, in nuclear physics where many nuclei can, abruptly, lose their stability [10, 29]. The growth of the role of the EPs may be also detected in the random-matrix ensembles with various interpretations [30] and in optical systems (where EPs are called degeneracies [31]). In classical magnetohydrodynamics Kato's exceptional points may even happen to lie *inside* the domain of acceptable parameters, separating merely the different dynamical regimes of the so-called α^2-dynamos [5].

In all these contexts, our present completion of our recent studies of EPs may find its future role and relevance as a classification tool offering a deeper geometric understanding of the structure of the domains $\mathcal{D}(H)$. Basically, our results seem to indicate an efficiency of a combination of the methods of algebra (e.g., of solvable equations) and analysis (offering, e.g., the optimal parametrizations of elementary curves and (hyper)surfaces) with the computer-assisted symbolic manipulations and with perturbation expansions. Perhaps, our explicit verification of the complementarity, compatibility and productivity of these methods could also lead, in a not too distant future, to the development of an explicit control of the stability of the systems, mediated by some purely algebraic tools of control of parameters in phenomenological quantum Hamiltonians.

Acknowledgments

Work supported by the GAČR grant nr 202/07/1307, by the MŠMT 'Doppler Institute' project nr LC06002 and by the Institutional Research Plan AV0Z10480505.

References

[1] Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **80** 5243
[2] Bender C M and Milton K A 1998 *Phys. Rev.* D **57** 3595
Andrianov A A, Cannata F, Dedonder J-P and Ioffe M V 1999 *Int. J. Mod. Phys.* A **14** 2675
Znojil M, Cannata F, Bagchi B and Roychoudhury R 2000 *Phys. Lett.* B **483** 284
Mostafazadeh A 2002 *Nucl. Phys.* B **640** 419
Znojil M 2002 *J. Phys. A: Math. Gen.* **35** 2341
Bagchi B, Mallik S and Quesne C 2002 *Mod. Phys. Lett.* A **17** 1651
Znojil M 2003 *Nucl. Phys.* B **662** 554
Lévai G 2004 *Czech. J. Phys.* **54** 1121
Samsonov B F and Shamshutdinova V V 2005 *J. Phys. A: Math. Gen.* **38** 4715
Bagchi B, Banerjee A, Caliceti E, Cannata F, Geyer H B, Quesne C and Znojil M 2005 *Int. J. Mod. Phys.* A **20** 7107
[3] Bender C M and Milton K A 1997 *Phys. Rev.* D **55** R3255
Kleefeld F 2005 *Czech. J. Phys.* **55** 1123
Jakubský V and Smejkal J 2006 *Czech. J. Phys.* **556** 985
Smejkal J, Jakubsky V and Znojil M 2006 Relativistic vector bosons and PT-symmetry *J. Phys. Studies* (*Preprint* hep-th/0611287) at press
[4] Mostafazadeh A 2003 *Class. Quantum Grav.* **20** 155
Mostafazadeh A 2004 *Czech. J. Phys.* **54** 93
Andrianov A A, Cannata F and Kamenschchik A Y 2006 *J. Phys. A: Math. Gen.* **39** 9975
[5] Günther U and Stefani F 2003 *J. Math. Phys.* **44** 3097
Guenther U, Stefani F and Znojil M 2005 *J. Math. Phys.* **46** 063504
Kirillov O and Günther U 2006 *J. Phys. A: Math. Gen.* **39** 10057
[6] Dorey P, Dunning C and Tateo R 2001 *J. Phys. A: Math. Gen.* **34** L391
Dorey P, Dunning C and Tateo R 2001 *J. Phys. A: Math. Gen.* **34** 5679
Dorey P, Dunning C and Tateo R 2004 *Czech. J. Phys.* **54** 35

[7] Bender C M, Boettcher S and Meisinger P N 1999 *J. Math. Phys.* **40** 2201
Fernández F M, Guardiola R, Ros J and Znojil M 1998 *J. Phys. A: Math. Gen.* **31** 10105
Cannata F, Junker G and Trost J 1998 *Phys. Lett.* A **246** 219
Delabaere E and Pham F 1998 *Phys. Lett.* A **250** 25
Gonzalez-Lopez A and Tanaka T 2006 *J. Phys. A: Math. Gen.* **39** 3715
Curthright T, Mezincescu L and Schuster D 2007 *J. Math. Phys.* **48** 092108 (*Preprint* quant-ph/0603170)
[8] Znojil M 1999 *Phys. Lett.* A **259** 220
Bagchi B and Roychoudhury R 2000 *J. Phys. A: Math. Gen.* **33** L1
Levai G and Znojil M 2000 *J. Phys. A: Math. Gen.* **33** 7165
Wang Q 2004 *Czech. J. Phys.* **54** 143
Mostafazadeh A and Ozcelik S 2006 *Turk. J. Phys.* **30** 437
Graefe E M and Korsch H J 2006 *Czech. J. Phys.* **56** 1007
[9] Bender C M, Brody D C and Jones H F 2002 *Phys. Rev. Lett.* **89** 0270401
Kretschmer R and Szymanowski L 2004 *Czech. J. Phys.* **54** 71
Mostafazadeh A 2004 *Czech. J. Phys.* **54** 1125
Znojil M 2004 *Rend. Circ. Mat. Palermo II* **72** (Suppl.) 211 (*Preprint* math-ph/0104012)
[10] Scholtz F G, Geyer H B and Hahne F J W 1992 *Ann. Phys. (NY)* **213** 74
[11] Znojil M (ed) 2004 *Pseudo-Hermitian Hamiltonians in Quantum Physics* I *Proc. Workshop (Prague, June 2003) Czech. J. Phys.* **54** (1)
Znojil M (ed) 2004 *Pseudo-Hermitian Hamiltonians in Quantum Physics II Proc. Workshop (Prague, June 2004) Czech. J. Phys.* **54** (10)
Znojil M (ed) 2005 *Pseudo-Hermitian Hamiltonians in Quantum Physics III Proc. Workshop (Istanbul, June 2005) Czech. J. Phys.* **55** (9)
Geyer H, Heiss D and Znojil M (ed) 2006 Pseudo-Hermitian Hamiltonians in Quantum Physics IV Proc. Workshop (Stellenbosch, November 2005) *J. Phys. A: Math. Gen.* **39** (32)
Znojil M (ed) 2006 *Pseudo-Hermitian Hamiltonians in Quantum Physics V Proc. Workshop (Bologna, July 2006) Czech. J. Phys.* **56** (9)
Fring A, Jones H and Znojil M 2007 Pseudo-Hermitian Hamiltonians in Quantum Physics VI: Proc. Workshop (London, July 2007)
[12] Bender C M 2007 *Rep. Prog. Phys.* **70** 947
[13] Buslaev V and Grecchi V 1993 *J. Phys. A: Math. Gen.* **26** 5541
[14] Znojil M and Geyer H B 2006 *Phys. Lett.* B **640** 52
[15] Znojil M and Geyer H B 2007 *Phys. Lett.* B **649** 494 (erratum)
[16] Znojil M 2007 *Phys. Lett.* B **647** 225
[17] Znojil M 2007 *Phys. Lett.* A **367** 300
[18] Znojil M 2007 *J. Phys. A: Math. Theor.* **40** 4863
[19] Znojil M 2007 *J. Phys. A: Math. Theor.* **40** 13131
[20] Znojil M 2007 *Phys. Lett.* B **650** 440
[21] Znojil M 2007 Broken spectral reflection symmetry in a solvable PT-symmetric model *Preprint* arXiv:0710.0457 submitted
[22] Landau L D and Lifshitz E M 1977 *Quantum Mechanics* (Oxford: Pergamon)
[23] Greiner W 1997 *Relativistic Quantum Mechanics—Wave Equations* (Berlin: Springer)
[24] Kato T 1966 *Perturbation Theory for linear Operators* (Berlin: Springer) pp 64
[25] Pauli W 1943 *Rev. Mod. Phys.* **15** 175
Sudarshan E C G 1961 *Phys. Rev.* **123** 2183
Nagy K L 1966 *State Vector Spaces with Indefinite Metric in Quantum Field Theory* (Budapest: Akademiai)
Lee T D and Wick G C 1969 *Nucl. Phys.* B **9** 209
Nakanishi N 1971 *Phys. Rev.* D **3** 811
Kleefeld F 2003 *AIP Conf. Proc.* **660** 325
Ramirez A and Mielnik B 2003 *Rev. Mex. Fis.* **49 S2** 130
[26] http://gemma.ujf.cas.cz/∼znojil/conf/index.html
[27] Hatano N and Nelson D R 1966 *Phys. Rev. Lett.* **77** 570
Hatano N and Nelson D R 1997 *Phys. Rev.* B **56** 8651
Feinberg J and Zee A 1999 *Phys. Rev.* E **59** 6433
Feinberg J and Zee A 1999 *Nucl. Phys.* B **552** 599
Trefethen L N *et al* 2001 *Commun. Pure Appl. Math.* **54** 595
[28] Dembowski C *et al* 2003 *Phys. Rev. Lett.* **90** 034101
Heiss D 2004 *Czech. J. Phys.* **54** 1091

[29] Geyer H B 2004 *Czech. J. Phys.* **54** 51
Swanson M S 2004 *J. Math. Phys.* **45** 585
Rotter I 2005 *Czech. J. Phys.* **55** 1167
Barrett B R, Stetcu I, Navratil P and Vary J P 2006 *J. Phys. A: Math. Gen.* **39** 9983
[30] Ahmed Z and Jain S R 2003 *Phys. Rev.* E **67** 045106(R)
Jain S R 2006 *Czech. J. Phys.* **56** 1021
Berry M V 2006 *J. Phys. A: Math. Gen.* **39** 10013
[31] Berry M V 2004 *Czech. J. Phys.* **54** 1039